Lecture 6
9/30

Hw 1 posted (due 10/7)

solve any 10 problems (problems from textbooks)

start early, use piazza and OO

Antonio's OO via zoom

The Principle of Induction

We informally defined $\mathbb{N}$ as the set $\{1, 2, 3, \ldots\}$

$\Longrightarrow$ Need more precise definition to prove statements about $\mathbb{N}$

Idea: To generate $\mathbb{N}$ as a subset of $\mathbb{R}$, begin with the number 1 (which we defined as the multiplicative identity in $\mathbb{R}$ when discussing Field Axioms).

Define 2 to equal $1+1$

3 to equal $2+1$

$\Longrightarrow$ Want $\mathbb{N}$ to be the subset of $\mathbb{R}$ obtained by beginning with 1 and successively adding 1

Def The set $\mathbb{N}$ of natural numbers is the intersection of all sets $S \subseteq \mathbb{R}$ that have the following two properties

1) $1 \in S$

2) If $x \in S$, $x+1 \in S$

Def (Natural Numbers)

The set of natural numbers $\mathbb{N}$ is the intersection of all sets $S \subseteq \mathbb{R}$ that have the following two properties

a) $1 \in S$

b) If $x \in S$ then $x+1 \in S$

Thm (principle of Induction)

For each natural number n, let $P(n)$ be a mathematical statement. If properties a) and b) below hold, then for each $n \in \mathbb{N}$ the statement $P(n)$ is true.

a) $P(1)$ is true

b) For $k \in \mathbb{N}$, if $P(k)$ true then $P(k+1)$ is true

Proof. Let $S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. By definition, $S \subseteq \mathbb{N}$. On the other hand, (a) and (b) here imply that S satisfies a) and b) in the definition above. Since $\mathbb{N}$ is the smallest such set, $\mathbb{N} \subseteq S$.

Therefore $S = \mathbb{N}$ and $P(n)$ is true for each $n \in \mathbb{N}$ $\square$

Induction Outline

Base Case: 1^{st} statement is true [check if $P(1)$ true]

Inductive Hypothesis: Assume $P(n)$ true for $n \leq k$

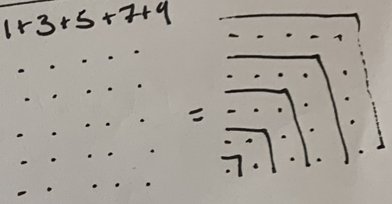
Inductive Step: show $k+1$ true

Conclusion: Every n is true

$$1 \to 2 \to 3 \to 4 \to \cdots$$

Another Ex

$n^2 = 1+3+5+7+9$

$5^2 =$ 

$= 1+3+5+7+9$

~~//////~~

Prove $1+3+5+7+9 = n^2$

Ex Prove $1+2+\cdots+n = \frac{n(n+1)}{2}$

Base: check $1 \overset{?}{=} \frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$ ✓

plug in a value and verify that LHS equals RHS

Inductive Hypothesis:

Assume $P(n)$ true $\forall n \leq k$

We assume $1+2+\cdots+n = \frac{n(n+1)}{2}$

Inductive Step:

Check / Verify whether replacing n with $k+1$ on the LHS equals replacing n with $k+1$ on the RHS

$$\underbrace{1+2+\cdots+k}_{\text{we assume this is } \frac{k(k+1)}{2}}+k+1 \overset{?}{=} \frac{(k+1)(k+1+1)}{2}$$

Does

$$\Rightarrow \frac{k(k+1)}{2}+k+1 \overset{?}{=} \frac{(k+1)(k+2)}{2}$$

$$k^2+k+2k+2 = k^2+k+2k+$$

Ex $11^n - 6$ is divisible by 5

Base: Check if $11^1 - 6$ is divisible by 5

$11 - 6 = 5$ and $5 | 5$ so yes ✓

Induction Hypothesis Assume $P(k)$ correct for some positive integer k

$\Longrightarrow 5 | 11^k - 6$

$\Longrightarrow \exists c$ s.t. $11^k - 6 = 5 \cdot c$

or $11^k = 5c + 6$

Inductive step Show that $P(k+1)$ is correct

Keep in mind what we know is true and what we want to show.

Want to show $11^{k+1} - 6$ can be expressed as a multiple of 5

Idea: start with $11^{k+1} - 6$ and rearrange to get something involving multiples of 5.

$\Longrightarrow$ At some point, use our assumption that $11^k = 5c + 6$

$11^{k+1} - 6 = 11 \times 11^k - 6$

$= 11(5c+6) - 6$ by induction hypothesis

$= 55c + 66 - 6$ expand bracket

$= 11(5c) + 60$

$= 5(11c + 12)$ since both have a common factor of 5 which is divisible by 5.

Ex $2^n > 2n \quad \forall n > 2$

Base: check $2^3 \overset{?}{>} 2 \cdot 3$

$8 > 6$ ✓

Inductive Hypothesis: Assume $2^k > 2 \cdot k$

Inductive Step: Verify that $2^{k+1} > 2 \cdot (k+1)$

Lets look at the LHS

$2^{k+1} = 2 \cdot 2^k$

by inductive hypothesis

$> 2 \cdot (2k)$ and $2 \cdot (2k) > 2(k+1) \; \forall k \geq 3$

$2k > k+1$

$> 2(k+1)$ ✓

Ex $2^n > n+4 \quad \forall n \geq 3$

Base $2^3 > 3+4$ hyp: $2^k > k+4$ for some $k \geq 4$

$8 > 7$ ✓

step $2^{k+1} > k+5$?

$2 \cdot 2^k \, \cancel{} - k - 5 > 0$?

$2 \cdot 2^k - k - 5 > 2(k+4) - k - 5$

$> k+3$ but since $k \geq 3$, $k+3 > 0$ ✓

Let $T_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$

$T_1 = 1$

$T_2 = 3$

Then for any $n \in \mathbb{N}$,

$$\cancel{\text{Prove}} \quad T_n + T_{n+1} = (n+1)^2$$

<u>Proof</u> <u>Base</u> $T_1 + T_2 = 1 + 3 = 4 \overset{?}{=} (1+1)^2 = 4$ ✓

$n=1$

Induction hyp. Let $K \in \mathbb{N}$, assume that

$$T_K + T_{K+1} = (K+1)^2$$

Step:

$$T_{K+1} + T_{K+2} \overset{?}{=} (K+1+1)^2$$

$\Longrightarrow$ Need to use the fact that since T_{K+1} is sum of first $K+1$ natural numbers, can write T_{K+1} as $T_K + (K+1)$

$$T_{K+1} + T_{K+2} = T_K + (K+1) + T_{K+1} + (K+2)$$

$$= \underbrace{T_K + T_{K+1}}_{} + 2K + 3$$

$$(K+1)^2 + 2K + 3$$

$$= K^2 + 2K + 1 + 2K + 3$$

$$= K^2 + 4K + 4 \overset{?}{=} (K+2)^2 \checkmark$$

Something about <u>all</u> natural numbers, but rather about this one natural number k. We thought the k^{th} domino knocking over the $(k+1)^{\text{st}}$.

Think of K as being a fixed natural number. It is an arbitrary choice so can be any natural number, but it is a single fixed choice. This induction hypothesis is not asserting

Ex For every $n \in \mathbb{N}$, the product of the first n odd natural numbers equals $\dfrac{(2n)!}{2^n \cdot n!}$

Def Factorial

The factorial of a positive integer n denoted $n!$ is defined to be

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

Ex $4! = 4\cdot 3\cdot 2\cdot 1$
$= 24$

i.e. $$1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{2^n\cdot n!}$$

Base $$1 \stackrel{?}{=} \frac{2\cdot 1}{2^1\cdot 1} = 1 \checkmark$$

Inductive hypothesis

Assume for some $K \in \mathbb{N}$,

$$1\cdot 3\cdot 5\cdots(2K-1) = \frac{(2K)!}{2^K\cdot K!}$$

Step: Verify $$1\cdot 3\cdot 5\cdot 7\cdots(2(K+1)-1) = \frac{(2(K+1))!}{2^{K+1}\cdot(K+1)!}$$

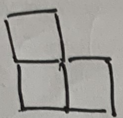
$$\Rightarrow \underbrace{1\cdot 3\cdot 5\cdots(2K-1)}(2K+1) = \frac{(2K+2)!}{2^{K+1}\cdot(K+1)!}$$

Replace term on LHS

$$\frac{(2K)!}{2^K\cdot K!}\cdot(2K+1) \stackrel{?}{=} \frac{(2K+1)!}{2^K\cdot K!} = \frac{(2K+1)!\cdot(2K+2)}{2^K\cdot K!\cdot(2K+2)}$$

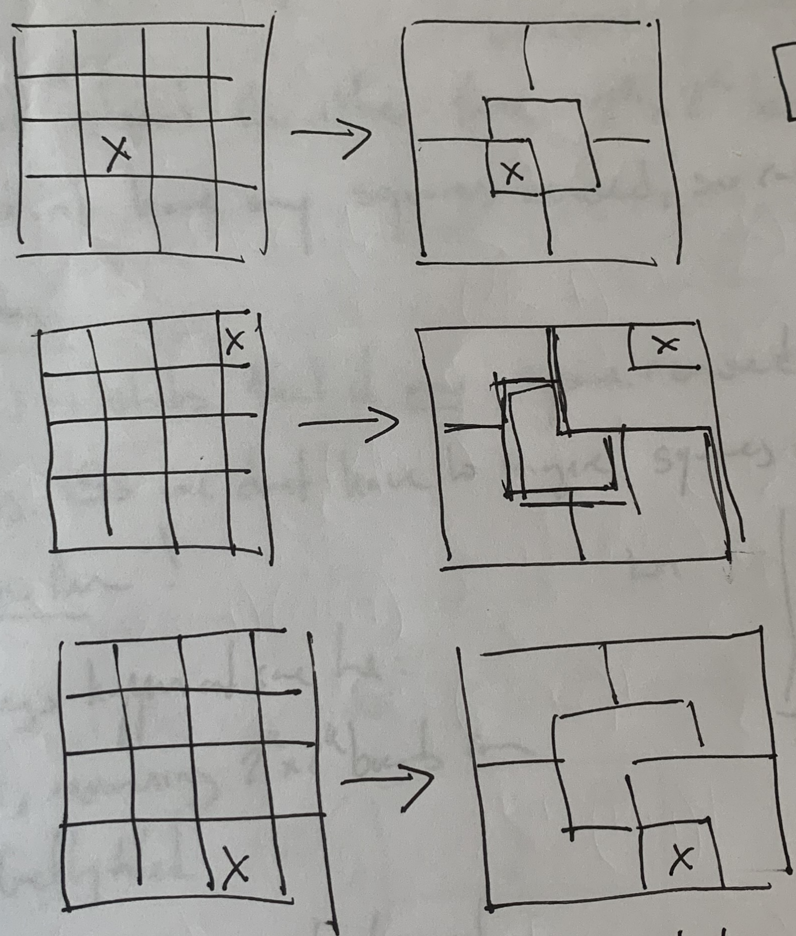
$$= \frac{(2K+2)!}{2^K\cdot K!\cdot 2(K+1)} = \frac{(2K+2)!}{2^{K+1}(K+1)!} \checkmark$$

Ex A tiling problem

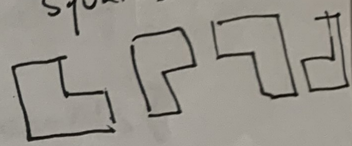
For every $n \in \mathbb{N}$, if any one square is removed from a $2^n \times 2^n$ chessboard, the result can be perfectly tiled with  -shaped tiles.

~~Proof Idea~~

Ex



Idea:
Tiles cover three squares

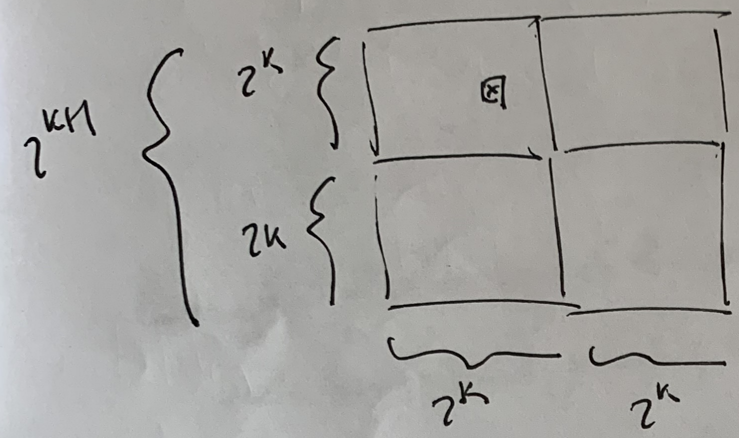


Can do trial and error with a small board, but a 64×64 board ~~or 2~~ not so easy.

Always ask: how are we going to use inductive hypothesis to prove this?

Inductive hypothesis deals with $2^k \times 2^k$ boards

<u>Ex</u> 4x4 chessboards inside 8x8 chessboards



There are four $2^k \times 2^k$ boards in the $2^{k+1} \times 2^{k+1}$ board.

One must contain the removed square and hence by inductive hypothesis be covered perfectly by tiles.

Q:

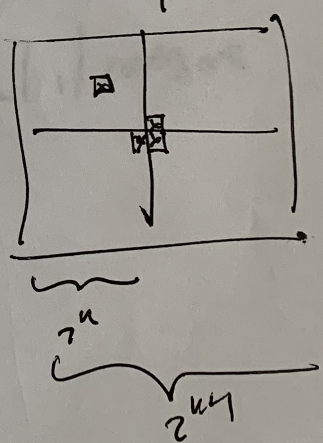
> But what about the other three $2^k \times 2^k$ boards?

They dont have any squares removed, so cant apply IH

Trick is to to them.

> Remember IH states that if <u>any</u> square removed, then perfect tiling exists. So we dont have to imagine squares randomly chosen, <u>we can choose them</u>!

Ex:



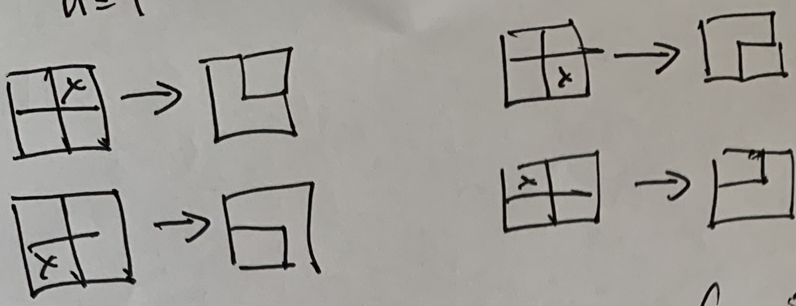
Then two things happen at same time:

1) by IH, remaining $2^k \times 2^k$ boards can be perfectly tiled

2) middle 3 squares are crossed out covered by a single tile.

Proof

Base $n=1$

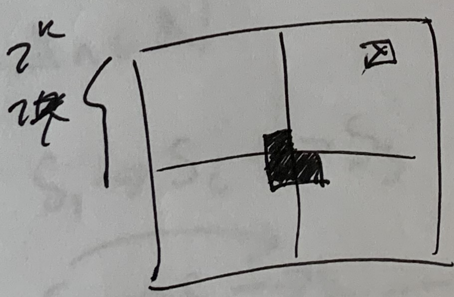


Among four possible squares that we can remove from $2^1\times 2^1$ board, each leaving board perfectly covered by a single shape.

IH: Let $k\in\mathbb{N}$ and assume if any one square is removed from a $2^k\times 2^k$ board, result can be perfectly covered

Step: Consider $2^{k+1}\times 2^{k+1}$ board w/ one square removed. Cut in half vertically and horizontally to form four $2^k\times 2^k$ boards. One of these four will have a square removed and hence by IH can be covered.

$\Rightarrow$ Next place a single -shaped tile so that it covers one square from each of the other three $2^k\times 2^k$ boards



Each of these other three $2^k\times 2^k$ boards can be perfectly covered by the IH so this $2^{k+1}\times 2^{k+1}$ can be covered $\square$